LECTURE 33 ANTIDERIVATIVES AND INTEGRATION

Linearity of Antidifferentiation

Example. Find the general antiderivative of the following functions:

(1) $f(x) = x^5$.
(2) $g(x) = \frac{1}{\sqrt{x}}$.
(3) $h(x) = \sin(2x)$.
(4) $i(x) = \cos\left(\frac{x}{2}\right)$.
(5) $j(x) = e^{-3x}$.
(6) $k(x) = 2^x$.

Antidifferentiation is a linear process (like differentiation), i.e. the antiderivative of $kf(x) \pm hg(x)$ with constants k, h is

$$kF(x) \pm hG(x) + C$$

where F and G are the antiderivatives of f and g respectively.

Example. Find the general antiderivative of

$$f(x) = \frac{3}{\sqrt{x}} + \sin(2x).$$

Solution. We utilize the fact that we can simply find the antiderivative of each component and add up. Define

$$g(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}, \quad h(x) = \sin(2x).$$

Then, we have

$$f(x) = 3g(x) + h(x) \implies F(x) = 3G(x) + H(x) + C$$

where G and H are antiderivatives of g and h respectively. All you need to make sure is that they are antiderivatives, not necessarily a general one, since the constant C is added already.

$$G(x) = 2x^{\frac{1}{2}}, \quad H(x) = -\frac{1}{2}\cos(2x)$$

and thus

$$F(x) = 6\sqrt{x} - \frac{1}{2}\cos(2x).$$

Initial Value Problems and Differential Equations

Finding an antiderivative for a function $f(x)$ is the same problem as finding a function $y(x)$ that satisfies the equation

$$\frac{dy}{dx} = f(x).$$

This is called a **differential equation**. The find $y(x)$, we must find the general antiderivative of f. This incurs a constant, which we can determine if we know a point of $y(x)$, usually specified as an **initial condition**

$$y(x_0) = y_0.$$

Combining the **differential equation** and the **initial condition**, we obtain an **initial value problem**. This type of problems appears all branches of science.

Example. A hot-air balloon ascending at the rate of 6 meters/sec is at a height 80 meters above the ground when a package is dropped. How long does it take the package to reach the ground?

Solution. Physics tells us that the package has an initial velocity of 6 meters/sec upwards the moment it is "dropped". Then, it goes on with free fall.

Let $v(t)$ be the velocity of the package at time t. Choose up as the positive direction. All we know is $v(0) = +6$.

However, the package is at free fall, which means it has a constant acceleration towards the earth, at $-9.8\ m^2/\text{s}$. Note also, that acceleration is the time derivative of velocity, $\frac{dv}{dt}$. Thus, we have the differential equation

$$\frac{dv}{dt} = -9.8$$

and an initial condition

$$v(0) = 6.$$

To solve the differential equation, we find the general antiderivative of the constant -9.8, which is

$$v(t) = -9.8t + C.$$

To determine C, we use the initial condition

$$6 = v(0) = -9.8 \cdot 0 + C \implies C = 6.$$

Therefore,

$$v(t) = -9.8t + 6.$$

Now, the question asks for when the package hits the ground, that is, when the position is at 0. Recall that velocity $v(t)$ is the time derivative of the position $s(t)$, which satisfies the differential equation,

$$\frac{ds}{dt} = v(t) = -9.8t + 6$$

with initial condition (we started 80 meters above ground)

$$s(0) = 80.$$

To find $s(t)$, we find the general antiderivative of $v(t)$,

$$s(t) = -4.9t^2 + 6t + D.$$

To find the constant D, we use the initial condition

$$80 = s(0) = -4.9(0)^2 + 6(0) + D \implies D = 80.$$

Therefore,

$$s(t) = -4.9t^2 + 6t + 80.$$

The package will hit the ground when $s(t) = 0$. So, we need to find the zero of $s(t)$. By the quadratic formula, we have

$$t = \frac{-6 \pm \sqrt{6^2 + 4 \cdot 4.9 \cdot 80}}{-9.8}$$

and find

$$t \approx 4.69898, \quad t \approx -3.47449.$$

We certainly reject the negative root. The package hits around after approximately 4.69898 seconds after it is dropped off from the rising balloon.

Indefinite Integral

Instead of asking "please find the antiderivative of f" everytime, we condense all that request into one simple yet historically significant symbol – the integral sign.

Definition. The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by

$$\int f(x)\, dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the intergral, and x is the **variable of integration**.

Example. Evaluate

$$\int \left(x^2 - 2x + 5\right) dx$$

Solution.

$$\int \left(x^2 - 2x + 5\right) dx = \frac{x^3}{3} - x^2 + 5x + C$$

Remark. Evaluating an indefinite integral is **the same** as finding the general antiderivative of the integrand.

This means, the integral also has the **linear** property, that is

$$\int \left(af\left(x\right) + bg\left(x\right)\right) dx = a \int f\left(x\right) + b \int g\left(x\right) dx,$$

you can divide and conquer, but only when the functions are added or subtracted.

There is NO such rule as

$$\int \left(f\left(x\right) g\left(x\right)\right) dx = \left(\int f\left(x\right) dx\right) \left(\int g\left(x\right) dx\right).$$

Area and Estimating with Finite Sums

There are various quests in classical geometry that involve formulas of areas and volumes of shapes. For example, the formula for the area of a circle came from Archimedes by inscribing polygons inside a circle. He computed the area of the polygons for each choice of number of sides and then found a pattern. He then took a limit as the number of sides goes to infinity. At the time, π is not known. Archimedes' formula for the circle is $A = \frac{1}{2}Cr$ where C is the circumference of the circle, and his proof relies on comparing the area of a circle (unknown at the time) to the area of a triangle with base C and height r. In modern day understanding, this area formula certainly works since $C = 2\pi r$.

So, why integration, now? What really is integration? Archimedes' approach to the area of the cirlce, in fact, involves the area of a polygon. The area of a polygon involves adding up lots of identical triangles. Each of these triangles is an isosceles, with sides equal to the radius r. As we increase the number of sides, the triangle's height and base changes while the side length is preserved to be r.

We are used to adding things up **discretely** as the objects we add are **countable**. Integration is a concept of adding things up, **continuously**. In other words, you are adding very little things up, each with infinitesimal (varying according to some law/function) size. This makes computing an area a special case of integration. Whenever you need to "add very little things up", you think of integration.

Example. Scientific processes involving integration.

(1) Area, volume, etc.
(2) Displacement of a continuously moving object with time-varying velocity.
(3) Energy of a continuously moving object with time-varying
(4) Mass of an object with spatially inhomogeneous density.
(5) Probability of an event characterized by a range of values governed by some continuous density function.

In the examples, we see that there is always something about "varing" quantities – it is captured by a function $f\left(x\right)$. Here, we consider a concrete example.

Example. Find the area below the graph of the function $f\left(x\right) = \sqrt{2 - x^2}$ by approximating with two vertical bars.